

GROUPOID C^* -ALGEBRAS WITH HAUSDORFF SPECTRUM

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ABSTRACT. Suppose G is a second countable, locally compact Hausdorff groupoid with abelian stabilizer subgroups and a Haar system. We provide necessary and sufficient conditions for the groupoid C^* -algebra to have Hausdorff spectrum. In particular we show that the spectrum of $C^*(G)$ is Hausdorff if and only if the stabilizers vary continuously with respect to the Fell topology, the orbit space $G^{(0)}/G$ is Hausdorff, and, given convergent sequences $\chi_i \rightarrow \chi$ and $\gamma_i \cdot \chi_i \rightarrow \omega$ in the dual stabilizer groupoid \widehat{S} where the $\gamma_i \in G$ act via conjugation, if χ and ω are elements of the same fiber then $\chi = \omega$.

INTRODUCTION

One of the reasons that C^* -algebras are so well studied is that they have a very deep representation theory. Understanding the spectrum or primitive ideal space of a C^* -algebra, and in particular the topology on these spaces, can reveal a great deal of information about the underlying algebra. For example, if a separable C^* -algebra A has Hausdorff spectrum \widehat{A} then A is naturally isomorphic to the section algebra of an upper-semicontinuous bundle over \widehat{A} such that each fiber of the bundle is isomorphic to the compact operators. The continuous trace C^* -algebras, which can be classified by a cohomology element, are then algebras with Hausdorff spectrum whose associated bundles are “locally trivial” in an appropriate sense [11, Chapter 5]. Given a class of C^* -algebras it is an interesting problem to characterize those algebras which have Hausdorff spectrum.

For example, in [15] the author proves the following result. Suppose we are given a transformation group (H, X) such that H is abelian and the group action satisfies any of the conditions in the Mackey-Glimm dichotomy [12]. Then the transformation group C^* -algebra will have Hausdorff spectrum if and only if the stabilizer subgroups of the action vary continuously with respect to the Fell topology and the orbit space X/H is Hausdorff. In this paper we would like to extend the work of [15] from transformation groups to groupoids. The most straightforward generalization is the conjecture that, given a groupoid G with abelian stabilizer subgroups which satisfies the conditions of the Mackey-Glimm dichotomy, the groupoid C^* -algebra will have Hausdorff spectrum if and only if the stabilizers vary continuously in G and $G^{(0)}/G$ is Hausdorff. Interestingly, we will show that this “naive” generalization fails and that characterizing the groupoid C^* -algebras with Hausdorff spectrum requires a third condition. Furthermore, the correct generalization, presented in Section 1 as Theorem 2, is in some ways stronger than the results of [15], even for transformation groups. We finish the paper by providing some further examples in Section 2. In addition, we also prove that, unlike the T_0 or T_1 case, in the Hausdorff case the spectrum cannot be studied using only the stabilizer subgroupoid.

Before we get started we should review some preliminary material. Throughout the paper we will let G denote a second countable, locally compact Hausdorff groupoid with a Haar system $\{\lambda_u\}$. We will use $G^{(0)}$ to denote the unit space, r to denote the range map, and s to denote the source map. We will let $S = \{\gamma \in G : s(\gamma) = r(\gamma)\}$ be the stabilizer, or isotropy, subgroupoid of G . Observe that on S the range and source maps are equal and that $r = s : S \rightarrow G^{(0)}$ gives S a bundle structure over $G^{(0)}$. Given $u \in G^{(0)}$ the fiber $S_u = r|_S^{-1}(u)$ is a group and is called the stabilizer subgroup at u . Since S is a closed subgroupoid of G , it is always second countable, locally compact, and Hausdorff. However, S will have a Haar system if and only if the stabilizers vary continuously. That is, if and only if the map $u \mapsto S_u$ is continuous with respect to the Fell topology on closed subsets of S [14, Lemma 1.3].

One of the primary examples of groupoids are those built from transformation groups. If a second countable locally compact Hausdorff group H acts on a second countable locally compact Hausdorff space X then we can form the transformation groupoid $H \ltimes X$ in the usual fashion. The properties of the transformation groupoid are closely tied to those of the group action. For instance, the orbit space $H \ltimes X^{(0)}/H \ltimes X$ is homeomorphic to the orbit space of the action X/H . Furthermore, the stabilizer groups S_X of $H \ltimes X$ can be naturally identified with the stabilizer subgroups H_x of H with respect to the group action and the stabilizers will vary continuously in $H \ltimes X$ if and only if they vary continuously in H .

Given a groupoid G we can construct the groupoid C^* -algebra $C^*(G)$ as a universal completion of the convolution algebra $C_c(G)$ [13, 5]. Of particular interest to us will be the spectrum $C^*(G)^\wedge$ of the groupoid algebra. One special case which will play a key role in our results is the spectrum of the stabilizer subgroupoid. Suppose that G has abelian stabilizer subgroups, that is, suppose the fibers of S are all abelian. If the stabilizers vary continuously so that S has a Haar system then we may construct the groupoid algebra $C^*(S)$. It turns out that in this case $C^*(S)$ is abelian and the spectrum of $C^*(S)$, denoted by \widehat{S} , is a second countable locally compact Hausdorff space which is naturally fibered over $G^{(0)}$. Furthermore the fiber of \widehat{S} over $u \in G^{(0)}$, which we will write as \widehat{S}_u , is the Pontryagin dual of the fiber S_u [8, Section 3]. We refer to \widehat{S} as the dual stabilizer groupoid. One of the things that makes \widehat{S} so useful is that its topology is relatively well understood; [8] gives a complete description of the convergent sequences in \widehat{S} . Since we will use this characterization quite a bit we have restated it below.

Proposition 1 ([8, Proposition 3.3]). *Suppose the groupoid G has continuously varying abelian stabilizers and that $\{\chi_n\}$ is a sequence in \widehat{S} with $\chi_n \in \widehat{S}_{u_n}$ for all n . Given $\chi \in \widehat{S}_u$ we have $\chi_n \rightarrow \chi$ if and only if*

- (a) $u_n \rightarrow u$ in $G^{(0)}$, and
- (b) given $s_n \in S_{u_n}$ for all n and $s \in S_u$ if $s_n \rightarrow s$ then $\chi_n(s_n) \rightarrow \chi(s)$.

The final thing we need to review is the notion of a groupoid action. A groupoid G can only act on spaces X which are fibered over $G^{(0)}$. If there is a surjective function $r_X : X \rightarrow G^{(0)}$ then we define a groupoid action via a map $\{(\gamma, x) : s(\gamma) = r_X(x)\} \rightarrow X$ such that for composable γ and η we have $\gamma \cdot (\eta \cdot x) = \gamma\eta \cdot x$. Among other things, this implies that $r_X(x) \cdot x = x$ for all $x \in X$ and $r_X(\gamma \cdot x) = r(\gamma)$. We will use the following three actions in this paper. Any groupoid G has actions on

its unit space $G^{(0)}$ and its stabilizer subgroupoid S which are defined as follows

$$\gamma \cdot u = \gamma u \gamma^{-1} = r(\gamma) \quad \text{on } G^{(0)}, \text{ and } \quad \gamma \cdot s = \gamma s \gamma^{-1} \quad \text{on } S.$$

Furthermore if S has abelian fibers which vary continuously then there is an action of G on \widehat{S} . For $\gamma \in G$, $\chi \in \widehat{S}_{s(\gamma)}$ we define

$$\gamma \cdot \chi(s) = \chi(\gamma^{-1} s \gamma) \quad \text{for } s \in S_{r(\gamma)}.$$

Given an action of G on a space X we will use $G \cdot x$ to denote the orbit of x in X and $[x]$ to denote the corresponding element of X/G . We would also like to recall that the orbit space X/G is locally compact, but not necessarily Hausdorff, and that the quotient map $q : X \rightarrow X/G$ is open as long as G has a Haar system [7, Lemma 2.1].

1. GROUPOID C^* -ALGEBRAS WITH HAUSDORFF SPECTRUM

As mentioned in the introduction, we would like to generalize the main result of [15], which has been restated below, from transformation groups to groupoids.

Theorem 1 ([15, Page 320]). *Suppose that (H, X) is an abelian transformation group and that the maps of H/H_x onto $H \cdot x$ are homeomorphisms for each $x \in X$. Then the spectrum of the transformation group C^* -algebra $C^*(H, X)$ is Hausdorff if and only if the map $x \mapsto H_x$ is continuous with respect to the Fell topology and X/H is Hausdorff.*

Remark 2. The condition that the maps of H/H_x onto $H \cdot x$ are homeomorphisms for each $x \in X$ is one of the equivalent conditions in the Mackey-Glimm dichotomy [12]. Following [2] we will refer to groupoids and transformation groups which satisfy one, and hence all, of the conditions of the Mackey-Glimm dichotomy as *regular*.

An important question is how to generalize the hypothesis that the group H is abelian. The most natural replacement is to assume that the stabilizer subgroups S_u are abelian for all $u \in G^{(0)}$. Since, as we will see, the regularity hypothesis can be removed completely, this leaves us with the following conjecture.

Conjecture. Suppose the groupoid G has abelian stabilizers. Then $C^*(G)$ will have Hausdorff spectrum if and only if the stabilizers vary continuously and $G^{(0)}/G$ is Hausdorff.

However, we will find that this conjecture fails and the assumption that G has abelian stabilizers is a weaker condition, even for transformation groups. Let us start by assuming G is a second countable, locally compact Hausdorff groupoid with abelian stabilizers and that $C^*(G)^\wedge$ is Hausdorff. It then follows from [8, Proposition 3.1] that the stabilizers must vary continuously. Next consider the following useful lemma.

Lemma 3. *Suppose G is a second countable locally compact Hausdorff groupoid with continuously varying abelian stabilizers. Then the following are equivalent:*

- (a) $C^*(G)$ has T_0 spectrum.
- (b) $C^*(G)$ is GCR.
- (c) $G^{(0)}/G$ is T_0 .

Furthermore, if any of these conditions hold then the map $[\gamma] \rightarrow r(\gamma)$ from G_u/S_u to $G \cdot u$ is a homeomorphism for all $u \in G^{(0)}$ and G is regular.

Proof. The groupoid algebra is separable since G is second countable. In this case the equivalence of the first two conditions follows from [9, Theorem 6.8.7]. Since the stabilizers are abelian, and therefore amenable and GCR, the equivalence of the second two conditions now follows from the main result of [1]. Finally, if $G^{(0)}/G$ is T_0 then it follows from [12] that the map $[\gamma] \mapsto r(\gamma)$ from G_u/S_u onto $G \cdot u$ is a homeomorphism for all $u \in G^{(0)}$ and hence G is regular in the sense of [2]. \square

Since we have assumed $C^*(G)^\wedge$ is Hausdorff, Lemma 3 implies that G is regular. We may now use [2, Theorem 3.5] to conclude that $C^*(G)^\wedge$ is homeomorphic to \widehat{S}/G . A brief argument shows that $G^{(0)}/G$ is homeomorphic to its image in \widehat{S}/G equipped with the relative topology. Thus $G^{(0)}/G$ is Hausdorff. This demonstrates one direction of our conjecture. On the other hand, suppose that G has continuously varying abelian stabilizers and that $G^{(0)}/G$ is Hausdorff. Then $G^{(0)}/G$ is certainly T_0 so that G is regular. It then follows from [2, Theorem 3.5] that $C^*(G)^\wedge$ is homeomorphic to \widehat{S}/G . So we will have proven our conjecture if we can show that \widehat{S}/G is Hausdorff. What is more, setting aside the issue of continuously varying stabilizers for the moment, we also have the following suggestive proposition.

Proposition 4. *Suppose G is a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilizers. Then $C^*(G)^\wedge$ is T_1 (resp. T_0) if and only if $G^{(0)}/G$ is T_1 (resp. T_0).*

Proof. It follows from Lemma 3 that $C^*(G)^\wedge$ is T_0 if and only if $G^{(0)}/G$ is. Now suppose $C^*(G)^\wedge$ is T_1 . Then Lemma 3 and [2, Theorem 3.5] imply that $C^*(G)^\wedge$ is homeomorphic to \widehat{S}/G . As noted above, $G^{(0)}/G$ is homeomorphic to its image in \widehat{S}/G , and as such $G^{(0)}/G$ is T_1 . Next suppose that $G^{(0)}/G$ is T_1 . Again using Lemma 3 and [2, Theorem 3.5] we have $C^*(G)^\wedge \cong \widehat{S}/G$. Thus, mirroring the Hausdorff case, we will be done if we can show that \widehat{S}/G is T_1 .

Suppose that we are given elements $[\rho], [\chi] \in \widehat{S}/G$ such that $[\rho] \neq [\chi]$. Let $p : \widehat{S} \rightarrow G^{(0)}$ be the bundle map and $\tilde{p} : \widehat{S}/G \rightarrow G^{(0)}/G$ its factorization. Set $[u] = \tilde{p}([\rho])$ and $[v] = \tilde{p}([\chi])$. Suppose $[u] \neq [v]$. Since $G^{(0)}/G$ is T_1 we can find open sets U and V such that $[u] \in U$, $[v] \in V$ and $[u] \notin V$, $[v] \notin U$. Then $\tilde{p}^{-1}(U)$ is an open set containing $[\rho]$ and not $[\chi]$ and $\tilde{p}^{-1}(V)$ is an open set containing $[\chi]$ and not $[\rho]$. Next suppose $[u] = [v]$. Since the fibers of S are abelian we have

$$(1) \quad s \cdot \chi(t) = \chi(s^{-1}ts) = \chi(t) \quad \text{for all } s \in S.$$

Hence the action of G on S is trivial when fixed to a single fiber and we can assume without loss of generality that $\rho, \chi \in \widehat{S}_u$ with $\rho \neq \chi$. Let $q : \widehat{S} \rightarrow \widehat{S}/G$ be the quotient map and recall that it is open. Fix a neighborhood U of ρ . If $\chi \notin G \cdot U$ then $[\chi] \notin q(U)$ and $q(U)$ separates $[\rho]$ from $[\chi]$. Now suppose $\chi \in G \cdot U$ for all neighborhoods U of ρ . Then for each U there exists $\gamma_U \in G$ and $\rho_U \in U$ such that $\rho_U = \gamma_U \cdot \chi$. If we direct ρ_U by decreasing U then it is clear that $\rho_U \rightarrow \rho$. This implies that $\gamma_U \cdot u = r(\gamma_U) = p(\rho_U) \rightarrow u$. Since G is regular $[\gamma] \mapsto r(\gamma)$ is a homeomorphism and we must have $[\gamma_U] \rightarrow [u]$ in G_u/S_u . However, the quotient map on G_u/S_u is open so that we may pass to a subnet, relabel, and choose $r_U \in S_u$ such that $\gamma_U r_U \rightarrow u$. Using (1)

$$\gamma_U r_U \cdot \chi = \gamma_U \cdot \chi = \rho_U \rightarrow u \cdot \chi = \chi.$$

Thus $\rho = \chi$, which is a contradiction. It follows that we must have been able to separate $[\rho]$ from $[\chi]$. This argument is completely symmetric so that we can also

find an open set around $[\chi]$ which does not contain $[\rho]$. It follows that \widehat{S}/G , and hence $C^*(G)^\wedge$, is T_1 . \square

The essential component of this proof is the argument that \widehat{S}/G is T_1 if $G^{(0)}/G$ is T_1 . We would like to extend this to the Hausdorff case but there are topological obstructions. We start by recalling Green's famous example of a free group action that is not proper.

Example 5 ([3]). The space $X \subset \mathbb{R}^3$ will consist of countably many orbits, with the points $x_0 = (0, 0, 0)$ and $x_n = (2^{-2n}, 0, 0)$ for $n \in \mathbb{N}$ as a family of representatives. The action of \mathbb{R} on X is described by defining maps $\phi_n : \mathbb{R} \rightarrow X$ such that $\phi_n(s) = s \cdot x_n$. In particular we let $\phi_0(s) = (0, s, 0)$ and for $n \geq 1$

$$\phi_n(s) = \begin{cases} (2^{-2n}, s, 0) & s \leq n \\ (2^{-2n} - (s - n)2^{-2n-1}, n \cos(\pi(s - n)), n \sin(\pi(s - n))) & n < s < n + 1 \\ (2^{-2n-1}, s - 1 - 2n, 0) & s \geq n + 1. \end{cases}$$

For instance, brief computations show that

$$(2) \quad 2n + 1 \cdot (2^{-2n}, 0, 0) = (2^{-2n-1}, 0, 0)$$

for all n . It is straightforward to observe that the orbit space X/\mathbb{R} is homeomorphic to the subset $\{x_n\}_{n=0}^\infty$ of \mathbb{R}^3 .

In the following we build an example of a transformation groupoid G with continuously varying abelian stabilizers such that $G^{(0)}/G$ is Hausdorff and \widehat{S}/G is not. This shows that, even in the transformation group case, our conjecture fails and that we cannot use the straightforward generalization of Theorem 1.

Example 6. Let \mathbb{R} act on X as in Example 5. Now restrict this action to the action of \mathbb{Z} on the subset $Y = \{\phi_n(m) : n \in \mathbb{N}, m \in \mathbb{Z}\}$. Let $H = \mathbb{Q}_D \rtimes_\phi \mathbb{Z}$ be the semidirect product, where \mathbb{Q}_D denotes the rationals equipped with the discrete topology and where we define

$$(3) \quad \phi(n)(r) = r2^n$$

for all $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$. It is easy to show that ϕ is a homomorphism from \mathbb{Z} into the automorphism group of \mathbb{Q}_D . Thus H is a locally compact Hausdorff group which is second countable because it is a countable discrete space. Recall that the group operations are given by

$$(q, n)(p, m) = (q + 2^n p, n + m) \quad (q, n)^{-1} = (-2^{-n} q, -n).$$

Let the second factor of H act on Y as in Example 5. In other words, let $(q, n) \cdot x := n \cdot x$. It is straightforward to show that this is a continuous group action. It follows that the transformation groupoid $G = H \ltimes Y$ is a second countable, locally compact Hausdorff groupoid with a Haar system. Furthermore, the stabilizer subgroup of H at x is $H_x = \{(q, 0) : q \in \mathbb{Q}\}$ for all $x \in Y$. Since $(q, 0)(r, 0) = (q + r2^0, 0) = (q + r, 0)$, the stabilizers are abelian, and since the stabilizers are also constant, they must vary continuously in both H and G . It will be important for us to observe that S is isomorphic to $\mathbb{Q}_D \times Y$ via the map $((q, 0), x) \mapsto (q, x)$. Finally, $\{x_n\}_{n=0}^\infty$ forms a set of representatives for the orbit space and it is not difficult to show that Y/G is actually homeomorphic to $\{x_n\}_{n=0}^\infty$ and is therefore Hausdorff.

To show that \widehat{S}/G is not Hausdorff we must first compute the dual. Since S is isomorphic to $\mathbb{Q}_D \times Y$ we can identify \widehat{S} with $\widehat{\mathbb{Q}_D \times Y}$. While $\widehat{\mathbb{Q}_D}$ is fairly mysterious

we do know that since $\hat{r}(s) = e^{irs}$ is a character on \mathbb{R} for all $r \in \mathbb{R}$ it must also be a character on \mathbb{Q}_D . Now suppose $((q, n), x) \in G$ and $(\hat{r}, -n \cdot x) \in \mathbb{Q}_D \times Y$. We have

$$\begin{aligned} ((q, n), x) \cdot (\hat{r}, -n \cdot x)(p, x) &= (\hat{r}, -n \cdot x)((q, n), x)^{-1}((p, 0), x)((q, n), x) \\ &= (\hat{r}, -n \cdot x)((-2^{-n}q, -n)(p, 0)(q, n), -n \cdot x) \\ &= (\hat{r}, -n \cdot x)((2^{-n}p, 0), -n \cdot x) \\ &= e^{irp2^{-n}} = (\widehat{2^{-n}r}, x)(p, x). \end{aligned}$$

Or, more succinctly,

$$(4) \quad ((q, n), x) \cdot (\hat{r}, -n \cdot x) = (\widehat{2^{-n}r}, x).$$

Next let $\gamma_n = ((0, 2n+1), (2^{-2n-1}, 0, 0))$ for all n . Using the inverse of (2) we have

$$r(\gamma_n) = (2^{-2n-1}, 0, 0) \quad \text{and} \quad s(\gamma_n) = (2^{-2n}, 0, 0).$$

If we set $\chi_n = (\hat{1}, (2^{-2n}, 0, 0))$ then clearly $\chi_n \rightarrow \chi = (\hat{1}, (0, 0, 0))$. Using (4) we compute $\gamma_n \cdot \chi_n = (\widehat{2^{-(2n+1)}}, (2^{-2n-1}, 0, 0))$. A quick calculation shows that $\gamma_n \cdot \chi_n \rightarrow \omega = (\hat{0}, (0, 0, 0))$. Hence $[\chi_n] \rightarrow [\chi]$ and $[\chi_n] \rightarrow [\omega]$. Since the action of G is trivial on fixed fibers this implies that \widehat{S}/G , and hence $C^*(G)^\wedge$, is not Hausdorff.

Even though our conjecture fails, we still know that if G has continuously varying abelian stabilizers and $G^{(0)}/G$ is Hausdorff then $C^*(G)^\wedge \cong \widehat{S}/G$. What we need is an additional hypothesis which, when taken in conjunction with $G^{(0)}/G$ being Hausdorff, will imply that \widehat{S}/G is Hausdorff. The appropriate condition is given below and forms the main result of the paper.

Theorem 2. *Suppose G is a second countable locally compact Hausdorff groupoid with a Haar system and abelian stabilizers. Then $C^*(G)$ has Hausdorff spectrum if and only if the following conditions hold:*

- (a) *the stabilizers vary continuously, i.e. $u \mapsto S_u$ is continuous with respect to the Fell topology,*
- (b) *the orbit space $G^{(0)}/G$ is Hausdorff, and,*
- (c) *given sequences $\{\chi_i\} \subset \widehat{S}$ and $\{\gamma_i\} \subset G$ with $\chi_i \in \widehat{S}_{s(\gamma_i)}$, if $\chi_i \rightarrow \chi$ and $\gamma_i \cdot \chi_i \rightarrow \omega$ such that χ and ω are in the same fiber then $\chi = \omega$.*

In essence the third condition prevents the kind of “looping” behavior we see in Example 6 and is enough to guarantee that \widehat{S}/G is Hausdorff.

Remark 7. Even in the case of transformation groups Theorem 2 is in some ways stronger than Theorem 1. The main advantage is that we only require the stabilizer groups to be abelian, and not the whole group. Furthermore, we also removed the regularity hypothesis. The price is that we have added a slightly technical condition that, while not easy to say, is simple enough to check in practice.

Proof. In the discussion following our conjecture at the beginning of the section on page 3 we showed that if $C^*(G)^\wedge$ is Hausdorff then conditions (a) and (b) hold and that \widehat{S}/G is Hausdorff. Now suppose we have $\chi_i \rightarrow \chi$ and $\gamma_i \cdot \chi_i \rightarrow \omega$ as in condition (c). Then $[\chi_i] \rightarrow [\chi]$ and $[\chi_i] \rightarrow [\omega]$. Since \widehat{S}/G is Hausdorff this implies $[\omega] = [\chi]$. However, χ and ω live in the same fiber and the action of G on a single fixed fiber is free so that $\chi = \omega$.

Now suppose conditions (a)-(c) are satisfied. Then again following the discussion on page 3, the first two conditions imply that $C^*(G)^\wedge$ is homeomorphic to \widehat{S}/G . Now suppose $[\chi_i] \rightarrow [\chi]$ and $[\chi_i] \rightarrow [\omega]$ in \widehat{S}/G . Using the fact that the quotient map is open we can pass to a subsequence, relabel, and choose new representatives χ_i so that $\chi_i \rightarrow \chi$. As before let $p : \widehat{S} \rightarrow G^{(0)}$ be the bundle map and let $\tilde{p} : \widehat{S}/G \rightarrow G^{(0)}/G$ be the natural factorization. Define $u_i = p(\chi_i)$ and $u = p(\chi)$ and observe that $[u_i] \rightarrow [u]$. Furthermore if $p(\omega) = v$ then $[u_i] \rightarrow [v]$ as well. Since $G^{(0)}/G$ is Hausdorff we have $[u] = [v]$ and we may assume, without loss of generality, that $u = v$. Now pass to a subsequence again, relabel, and find $\gamma_i \in G$ such that $\gamma_i \cdot \chi_i \rightarrow \omega$. These sequences satisfy the hypothesis of (c) so $\omega = \chi$. It follows $[\omega] = [\chi]$ and that \widehat{S}/G , and hence $C^*(G)^\wedge$, is Hausdorff. \square

It should be noted that there are a variety of situations in which condition (c) is guaranteed to hold.

Proposition 8. *Let G be a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilizers. Then condition (c) of Theorem 2 automatically holds if G satisfies any of the following:*

- (a) $G = H \ltimes X$ is an abelian transformation groupoid,
- (b) G is principal,
- (c) G is proper,
- (d) G is Cartan, or
- (e) G is transitive.

Proof. Let $\chi_i \rightarrow \chi$ and $\gamma_i \cdot \chi_i \rightarrow \omega$ be as in condition (c). Set $u_i = s(\gamma_i)$, $v_i = r(\gamma_i)$, $u = p(\chi) = p(\omega)$ and observe that $u_i \rightarrow u$ and $v_i \rightarrow u$. Now suppose $G = H \ltimes X$ where H is abelian. Then we must have $\gamma_i = (t_i, v_i)$ with $u_i = t_i^{-1} \cdot v_i$. Given s in the stabilizer subgroup H_u we can use the fact that the stabilizers vary continuously to pass to a subsequence, relabel, and find $s_i \in H_{u_i}$ such that $s_i \rightarrow s$ in H . Consequently $(s_i, u_i) \rightarrow (s, u)$ and by Proposition 1 $\chi_i(s_i, u_i) \rightarrow \chi(s, u)$. On the other hand, since the group is abelian, we also have $s_i \in H_{v_i} = H_{t_i \cdot u_i}$ for all i . It follows that $(s_i, v_i) \rightarrow (s, u)$ in S and therefore

$$(t_i, v_i) \cdot \chi_i(s_i, v_i) = \chi_i(t_i^{-1} s_i t_i, u_i) = \chi_i(s_i, u_i) \rightarrow \omega(s, u).$$

Hence $\chi = \omega$ and condition (c) automatically holds for abelian transformation groups.

Moving on, condition (c) trivially holds if G is principal. For the next two conditions observe the following. Suppose we can pass to a subsequence, relabel, and find $\gamma \in G$ such that $\gamma_i \rightarrow \gamma$. It follows that $\gamma_i \cdot \chi_i \rightarrow \gamma \cdot \chi$ and therefore $\gamma \cdot \chi = \omega$. However, the range and source maps are continuous so we must have $r(\gamma) = s(\gamma) = u$ and hence $\gamma \in S_u$. The fibres of S are abelian so that by (1) $\omega = \gamma \cdot \chi = \chi$. Thus it will suffice to show that we can prove γ_i has a convergent subsequence. However, if G is either proper or Cartan then this follows almost by definition.

Finally, suppose G is transitive. Since G is also second countable [6, Theorem 2.2] implies that the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is open. Thus we can pass to a subsequence, relabel, and find $\eta_i \in G$ such that $r(\eta_i) = v_i$, $s(\eta_i) = u_i$ and $\eta_i \rightarrow \eta$. Observe that $\eta_i^{-1} \gamma_i \in S_{u_i}$ for all i so that $\gamma_i \cdot \chi_i = \eta_i \cdot (\eta_i^{-1} \gamma_i \cdot \chi_i) = \eta_i \cdot \chi_i$. Thus $\gamma_i \cdot \chi_i = \eta_i \cdot \chi_i \rightarrow \eta \cdot \chi = \chi$. It follows that $\chi = \omega$ and condition (c) holds in this case as well. \square

2. EXAMPLES AND DUALITY

In this section we would like to begin by applying Theorem 2 to several examples.

Example 9. Let $H = SO(3, \mathbb{R})$, $X = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and let H act on X by rotation. It is clear that H is not abelian, and therefore we cannot apply Theorem 1. However, it does have abelian stabilizer subgroups. Given a vector $v \in X$ it's easy to see that S_v is the set of rotations about the line described by v . In particular, this is isomorphic to the circle group and is therefore abelian. What is more, some computations show that the stabilizers vary continuously and that the stabilizer subgroupoid S is homeomorphic to $X \times \mathbb{T}$. This in turn implies that the dual groupoid is homeomorphic to $X \times \mathbb{Z}$. Now suppose $(U_i^{-1}v_i, \chi_i) \rightarrow (v, \chi)$ and $(U_i, v_i) \cdot (U_i^{-1}v_i, \chi_i) \rightarrow (v, \omega)$ as in condition (c). Given $\theta_i \rightarrow \theta$ in \mathbb{T} we have from Proposition 1 that

$$(U_i^{-1}v_i, \chi_i)(U_i^{-1}v_i, \theta_i) = \chi_i(\theta_i) \rightarrow (v, \chi)(v, \theta) = \chi(\theta).$$

Using the fact that conjugating rotation about an axis w by $V \in H$ gives us the corresponding rotation about Vw , we also have

$$(U_i, v_i) \cdot (U_i^{-1}v_i, \chi_i)(v_i, \theta_i) = (U_i^{-1}v_i, \chi_i)(U_i^{-1}v_i, \theta_i) = \chi_i(\theta_i) \rightarrow (v, \omega)(v, \theta) = \omega(\theta).$$

It follows that $\chi = \omega$ and condition (c) of Theorem 2 holds. Finally, the orbit space X/H is homeomorphic to the open half-line and is therefore Hausdorff. Thus we can conclude that $C^*(H \ltimes X)$ has Hausdorff spectrum. In fact [2, Theorem 3.5] shows that $C^*(H \ltimes X)$ is homeomorphic to $\widehat{S}/H \ltimes X = (0, \infty) \times \mathbb{Z}$.

Example 10. Let E be a row finite directed graph with no sources. Recall that we can build the graph groupoid G as in [4]. Elements of G are triples (x, n, y) where x and y are infinite paths which are shift equivalent with lag n , and elements of $G^{(0)}$ are infinite paths.¹ Furthermore, the groupoid C^* -algebra $C^*(G)$ is isomorphic to the graph C^* -algebra. Let us consider the conditions of Theorem 2. First, the stabilizers are all subgroups of \mathbb{Z} and hence abelian. Furthermore, the groupoid G will have nontrivial stabilizers if and only if there exists an infinite path which is shift equivalent to itself. In other words, if and only if there is a cycle. Suppose a cycle on the graph has an entry. Let x be the path created by following the cycle an infinite number of times. For each $i \in \mathbb{N}$ let x_i be the path which, at its head, follows the cycle i times and then has a non-cyclic tail leading off from the entry. Because x_i eventually agrees with x on any finite segment we have $x_i \rightarrow x$. However none of the x_i are cycles so that S_{x_i} is trivial for all i . On the other hand $S_x \cong n\mathbb{Z}$ where n is the length of the cycle. Thus the stabilizers do not vary continuously. This shows that in order for the stabilizers to vary continuously no cycles in the graph can have entries. A similar argument shows that the converse holds as well.

For the second condition we require that the orbit space $G^{(0)}/G$ be Hausdorff. In this case the orbit space is the space of shift equivalence classes. Recall that the basic open sets in $G^{(0)}$ are the cylinder sets V_a . More specifically, a is a finite path and V_a is the set of all infinite paths which are initially equal to a . Given $[x] \in G^{(0)}/G$ we will have $x \in G \cdot V_a$ if and only if x is shift equivalent to a path with initial segment a . This is equivalent to there being a path from any vertex on x to the source of a . Conversely, $y \notin G \cdot V_a$ if and only if there is no path from any vertex on y to the source of a . Using these facts it follows from a brief argument

¹We will be using the Raeburn convention for path composition [10].

that $G^{(0)}/G$ will be Hausdorff if and only if given non-shift equivalent paths x and y there exists vertices u and v such that there is a path from a vertex on x to u , a path from a vertex on y to v , and there is no vertex w which has a path to both u and v .

Finally, for the third condition we observe that given $(y, n, x) \in G$, $(y, m, y) \in S$ and $\chi \in \widehat{S}_x$ we have

$$(5) \quad (y, n, x) \cdot \chi(y, m, y) = \chi((x, -n, y)(y, m, y)(y, n, x)) = \chi(x, m, x).$$

Now suppose $\chi_i \rightarrow \chi$ and $(y_i, n_i, x_i) \cdot \chi_i \rightarrow \omega$ in \widehat{S} with $\chi, \omega \in \widehat{S}_x$. Notice that this implies that we must have $x_i \rightarrow x$ and $y_i \rightarrow x$ in $G^{(0)}$. Let $(x, n, x) \in S_x$. Then $(x_i, n, x_i) \rightarrow (x, n, x)$ and by Proposition 1 $\chi_i(x_i, n, x_i) \rightarrow \chi(x, n, x)$. On the other hand we also know $(y_i, n, y_i) \rightarrow (x, n, x)$ so that, using (5) and Proposition 1,

$$(y_i, n_i, x_i) \cdot \chi_i(y_i, n, y_i) = \chi_i(x_i, n, x_i) \rightarrow \omega(x, n, x).$$

This implies that $\chi(x, n, x) = \omega(x, n, x)$. Hence $\chi = \omega$ and condition (c) is automatically satisfied. Put together this shows that the graph groupoid algebra, and therefore the graph algebra, will have Hausdorff spectrum if and only if

- no cycle has an entry and,
- given non-shift equivalent paths x and y we can find vertices u and v such that there is a path from a vertex on x to u , a path from a vertex on y to v , and there is no vertex w which has a path to both u and v .

One annoyance of Theorem 2 is that condition (c) requires us to deal with the dual stabilizer groupoid. Using the same technique as the proof of Theorem 2 one can show that if $G^{(0)}/G$ is Hausdorff and if condition (c) holds for sequences in S (not \widehat{S}) then S/G is Hausdorff. This raises the question of whether \widehat{S}/G is Hausdorff if and only if S/G is Hausdorff, which is interesting in its own right. Similar to the previous section we find that this question can be answered in the affirmative in the T_0 and T_1 cases. More specifically, using the topological argument given in Proposition 4, one can prove the following result.

Proposition 11. *Let G be a second countable, locally compact Hausdorff groupoid with continuously varying abelian stabilizers. Then either $G^{(0)}/G$, S/G , and \widehat{S}/G are all T_1 (resp. T_0) or none of them is T_1 (resp. T_0).*

Unfortunately, again similar to the previous section, this proposition doesn't extend to the Hausdorff case either, as we demonstrate below. This example also shows that is is not enough to verify (c) on S and that working with the dual is necessary.

Example 12. Let H , Y and G be as in Example 6. Recall that we have already shown that in this case \widehat{S}/G is not Hausdorff. The computations from Example 6 also show that condition (c) does not hold on \widehat{S} . Now we will show that S/G is Hausdorff and that S does satisfy condition (c). First, given $((q, n), y) \in G$ and $(r, x) \in S$ a computation similar to the one preceding (4) shows that

$$(6) \quad ((q, n), y) \cdot (r, x) = (r2^n, y).$$

Suppose $[s_i] \rightarrow [s]$ and $[s_i] \rightarrow [t]$ in S/G . Since Y/G is Hausdorff we can follow the same argument given in Theorem 2 to pass to subsequences, choose new representatives, and find $\gamma_i \in G$ so that $s_i \rightarrow s$ and $\gamma_i \cdot s_i \rightarrow t$ where $s, t \in S_u$. In particular this implies $s = (r, u)$ and $t = (q, u)$ for $r, q \in \mathbb{Q}$. Suppose $s_i = (r_i, x_i)$

and $\gamma_i = ((p_i, n_i), y_i)$. Then it follows from (6) that $\gamma_i \cdot s_i = (r_i 2^{n_i}, y_i)$. Hence $r_i \rightarrow r$ and $r_i 2^{n_i} \rightarrow q$. However, we gave \mathbb{Q}_D the discrete topology so that, eventually, $q = 2^{n_i} r_i = 2^{n_i} r$. Now, if either $r = 0$ or $q = 0$ then $s = t$. If $r, q \neq 0$ we know that eventually $n_i = n = \log_2(q/r)$. We may as well pass to a subnet and assume this is always true. But then $n_i \cdot x_i \rightarrow n \cdot x$. However, we also have $n_i \cdot x_i = \gamma_i \cdot x_i = y_i \rightarrow x$. Thus $n \cdot x = x$. But the action of \mathbb{Z} is free which implies $n = 0$. Thus $\log_2(q/r) = 0$ and $q = r$. It follows that $s = t$ and that S/G is Hausdorff. What is more, the above argument also shows that condition (c) holds for sequences in S .

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